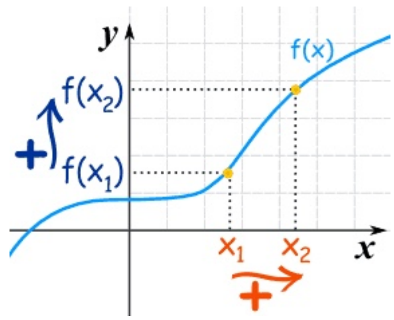
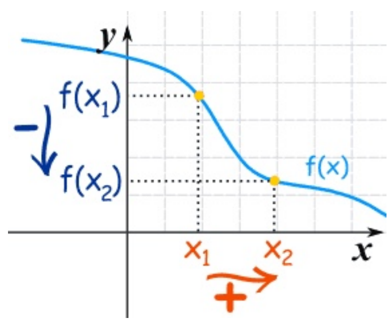
⊛ Increasing & Decreasing function over an interval I.

- Increasing $x_1, x_2 \in I$ s.t $x_1 < x_2 \Rightarrow f(x_1) \leqslant f(x_2)$
- Strictly increasing $x_1, x_2 \in I$ s.t $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

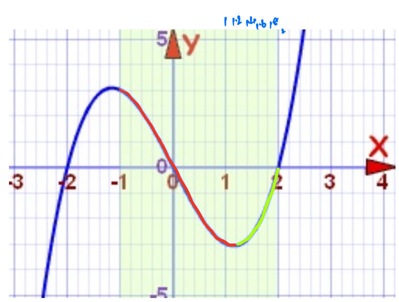


- Decreasing $x_1, x_2 \in I$ s.t $x_1 < x_2 \Rightarrow f(x_1) \geqslant f(x_2)$
- Strictly decreasing $x_1, x_2 \in I$ s.t $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



- An example to find where the following function is increasing or decreasing.

$f(x) = x^3 - 4x$, for $x \in [-1, 2]$



decreasing in $[-1, 1.2]$

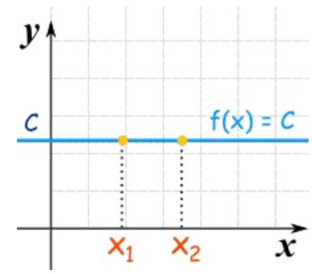
increasing in $[1.2, 2]$

Eg. (1) Constant functions.

For a constant (fixed) c.

$y = f(x) = c$, for all real x

So, $f(x_1) = c$, $f(x_2) = c$



$x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$ — neither strictly increasing / neither strictly decreasing

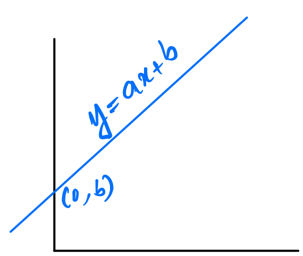
Eg. (2) Straight Lines:

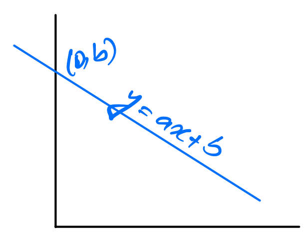
An equation of line can be given by $y = ax + b$. (a: slope)

For a straight line increasing/decreasing/constant is determined by the slope.

When $a = 0$: $y = 0x + b \Rightarrow y = b \longleftrightarrow$ Eg. 1

When $a \neq 0$: $a > 0$ $\qquad$ $a < 0$





⊛ One-one functions / Injective functions.

For any two values x_1, x_2 in I such that $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. ⊛ Note strictly increasing / decreasing functions

Eg. Check the following functions are one-one or not!!

① $f(x) = \sqrt{x^2+7}$, $x \in \mathbb{R}$

② $g(x) = x^3 - 8$, $x \in \mathbb{R}$.

① Note, if $x_1 = 1$, $x_2 = -1$,

then $x_1^2 = x_2^2 \Rightarrow x_1^2 + 7 = x_2^2 + 7 \Rightarrow \sqrt{x_1^2+7} = \sqrt{x_2^2+7}$

$\Rightarrow f(x_1) = f(x_2) \rightarrow$ f is not one-one

② $g(x) = x^3 - 8$

Note: the degree of $g(x)$ is 3 (an odd number), in this case there is no $x_1 \neq x_2$ in $\mathbb{R}$ for which $x_1^3 = x_2^3$ will hold. In other words,

$x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3$ for any choice of x_1 & x_2.

Then, let $x_1 \neq x_2$.

So, $x_1^3 \neq x_2^3 \Rightarrow x_1^3 - 8 \neq x_2^3 - 8 \Rightarrow g(x_1) \neq g(x_2)$

Therefore, $g(x)$ is an one-one function over all real numbers.

⊛ Composition & Combining functions.

Composition: If we have such situation, we have

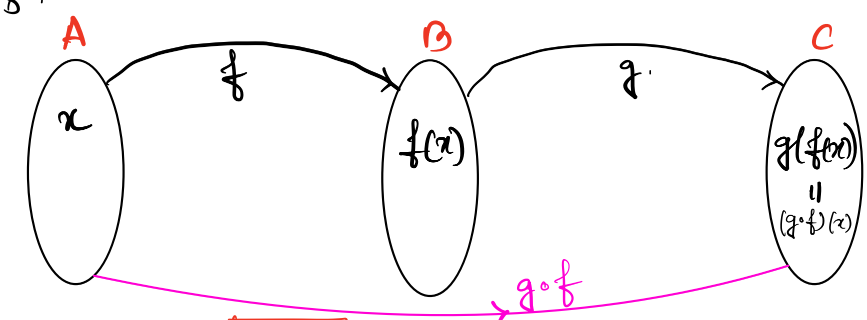
$f: A \longrightarrow B$ is a function & $g: B \longrightarrow C$ is a function

such that $\text{range}(f) \subseteq \text{domain}(g)$.

subset

Then can we define a single function between A & C!!

Yes, we can. It's called the composition of two functions.



Written by $\boxed{g \circ f}$ (said, g compose f)

Note: ① f gets the values first from x

② g gets the values then from f(x).

Therefore, domain of $g \circ f$ is same as the domain of f

So, $(g \circ f)(x) = g(f(x))$, domain of $g \circ f$ = dom. of f

Similarly, $(f \circ g)(x) = f(g(x))$, domain of $f \circ g$ = dom. of g

<u>Note</u>: $(g \circ f)(x) \neq (f \circ g)(x)$ in general.

<u>Eg</u>. $f(x) = 3x+1$, $x \in \mathbb{R}$

$g(x) = x^2 - 7$, $x \in \mathbb{R}$

$$(f \circ g)(x) = f(g(x))$$
$$= f(x^2-7)$$
$$= 3(x^2-7)+1$$
$$= 3x^2 - 21 + 1$$
$$= 3x^2 - 20$$

$(f \circ g)(x) \neq (g \circ f)(x)$

$$(g \circ f)(x) = g(f(x))$$
$$= g(3x+1)$$
$$= (3x+1)^2 - 7$$
$$= (3x+1)(3x+1) - 7$$
$$= (3x)(3x) + (3x).1 + 1.(3x) + 1.1 - 7$$
$$= 9x^2 + 3x + 3x + 1 - 7$$
$$= 9x^2 + 6x - 6$$

⊛ Algebra of Functions:

$(f+g)(x) = f(x) + g(x)$ ⟶ Sum

$(f-g)(x) = f(x) - g(x)$ ⟶ Difference

$(f \cdot g)(x) = f(x) \cdot g(x)$ ⟶ Product

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for $g(x) \neq 0$ ⟶ Quotient

⊛ $(f \cdot g)(x)$ & $(f \circ g)(x)$ are different completely.

$f(x) \cdot g(x)$ $\qquad$ $f(g(x))$

Be Careful

⊛ Even Functions & Odd Functions:

Even: $f(x) = f(-x)$, for all x in the domain.

Odd: $f(-x) = -f(x)$, for all x in the domain.

Eg. $f(x) = x^n$.

When n is even, $f(x) = f(-x)$, ie, f is even.

When n is odd, $f(-x) = -f(x)$, ie, f is odd.

Say, $f(x) = x^3 + x + 1$

⊛ replace x by $-x$ & evaluate.

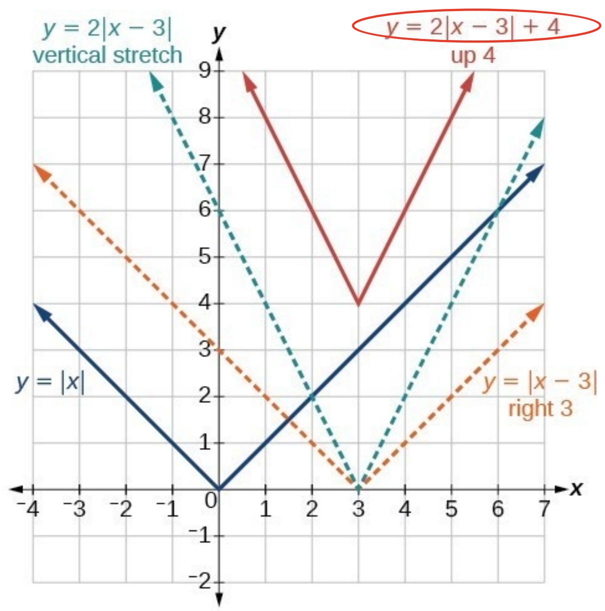
$$f(-x) = (-x)^3 + (-x) + 1 = (-x)(-x)(-x) + (-x)+1$$
$$= -(x \cdot x \cdot x) - x + 1$$
$$= -x^3 - x + 1$$

$$-f(x) = -(x^3 + x + 1) = -x^3 - x - 1$$

So, f is not odd function; not even function.

⊛ Absolute Value Functions:

$$f(x) = |x| \text{ or } abs(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$



$$y = |x-3|$$
$$= \begin{cases} -(x-3), & x-3 < 0 \\ x-3, & x-3 \geq 0 \end{cases}$$
$$= \begin{cases} -(x)+3, & x < 3 \\ x-3, & x \geq 3 \end{cases}$$

Domain of $y = 2|x-3|+4$ is $(-\infty, \infty)$ or $\{x \mid x \in \mathbb{R}\}$

Range: Note $|x-3| \geq 0$, for all x

$\Rightarrow 2|x-3| \geq 0$, for all x

$\Rightarrow 2|x-3|+4 \geq 4$, for all x

So, Range is $[4, \infty)$

⊛ Slope-Intercept form of a line passing through pts (x_1, y_1) & (x_2, y_2)

$$y = \underset{\substack{\uparrow \\ \text{slope}}}{m}x + \underset{\substack{\uparrow \\ y\text{-intercept } (0,b)}}{b}$$

slope formula : $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Standard line equation : $ax + by = c$, a & $b \neq 0$

⊛ Polynomials:

Polynomial function : $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
for some $n \geq 0$.
with $a_n \neq 0$

If $n=2$ we say quadratic function
$n=3$ we say cubic function

degree of polynomial = n

leading coefficient = a_n

Eg. $f(x) = x^2 + 6x + 9 \rightsquigarrow$ Quadratic , leading coeff = 1.

$f_1(x) = 100x^{47} + 8 \rightsquigarrow \deg(f_1) = 47$, leading coeff = 100

⊛ Roots of Quadratic $ax^2 + bx + c = 0$ given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve for x, ① $x^2+3x+1=0$
② $x^2+3x+7=0$

① $ax^2+bx+c \equiv x^2+3x+1 \Rightarrow a=1,\ b=3,\ c=1$

So, $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-3 \pm \sqrt{(3)^2-4(1)(1)}}{2(1)}$

$= \dfrac{-3 \pm \sqrt{9-4}}{2}$

$= \dfrac{-3 \pm \sqrt{5}}{2}$

$x = \dfrac{-3+\sqrt{5}}{2}$ or $\dfrac{-3-\sqrt{5}}{2}$

are solutions.

② $ax^2+bx+c \equiv x^2+3x+7 \Rightarrow a=1,\ b=3,\ c=7$

So, $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{-3 \pm \sqrt{(3)^2-4(1)(7)}}{2(1)}$

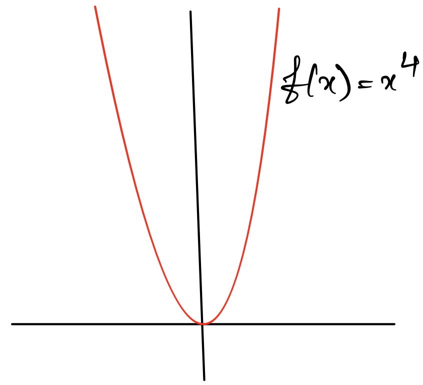
$= \dfrac{-3 \pm \sqrt{9-28}}{2}$

$= \dfrac{-3 \pm \sqrt{-19}}{2}$ → DNE

There is no real solution.

⊛ Power Functions: $f(x) = ax^b$.

Eg. $f(x) = x^4$



Behaviour at $\pm\infty$.

⊛ Algebraic Functions: rational form: $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$

root form: $f(x) = \sqrt{p(x)}$, $p(x) \geq 0$

Eg ① $f(x) = \frac{3x-1}{5x+2}$. Domain, Range

$5x+2 \neq 0$ $\quad x \neq -\frac{2}{5}$ $\quad y \neq \frac{3}{5}$

② $f(x) = \sqrt{4-x^2}$. Domain, Range

$4-x^2 \geq 0$

$\Rightarrow x^2 \leq 4$

$\Rightarrow -2 \leq x \leq 2$

$[0, 2]$

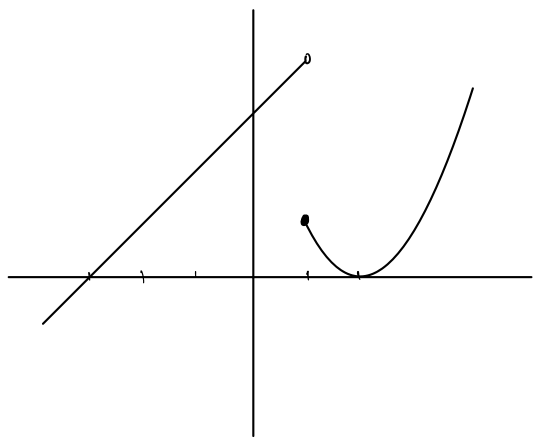
⊛ Transcendental Functions:

- trigonometric
- exponential
- logarithmic

} most common.

⊛ Piecewise defined functions:

$$f(x) = \begin{cases} x+3, & x < 1 \\ (x-2)^2, & x \geq 1 \end{cases}$$



Transformation of $f(c > 0)$	Effect on the graph of f
$f(x) + c$	Vertical shift up c units
$f(x) - c$	Vertical shift down c units
$f(x + c)$	Shift left by c units
$f(x - c)$	Shift right by c units
$cf(x)$	Vertical stretch if $c > 1$; vertical compression if $0 < c < 1$
$f(cx)$	Horizontal stretch if $0 < c < 1$; horizontal compression if $c > 1$
$-f(x)$	Reflection about the x-axis
$f(-x)$	Reflection about the y-axis