Continuity:

Handwavy understanding is, it's about being able to draw graph continuously without lifting the pen/pencil from the paper.

Def^n:- A function $f(x)$ is continuous at $x=a$ if & only if

(i) $f(a)$ is defined

(ii) $\lim_{x\to a} f(x)$ exists

(iii) $\lim_{x\to a} f(x) = f(a)$

} satisfies.

⊛ If it fails to be continuous, we say it's discontinuous at $x=a$.

⊛ f is continuous on an interval if f is continuous at every point a in that interval.

Determining Continuity at a point:

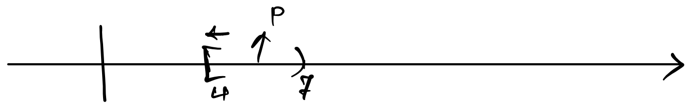
$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Then $f(0)=1$ & we have seen $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Therefore, $f(x)$ is continuous at $x=0$

Determing Continuity over an Interval:

$f(x) = \sqrt{x-4}$, x is in $[4, 7)$



<u>Note</u>: $[4, 7)$ means domain has the point $x=4$ in it, but not 7.

So we need to check for two cases:

<u>Case</u> 1: Continuity at $x=4$

We need to check for $\lim_{x \to 4^+} f(x)$ & $f(4)$ only.

Now, $f(4) = \sqrt{4-4} = \sqrt{0} = 0$

& $\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$

So, $\lim_{x \to 4^+} f(x)$ exists & $\lim_{x \to 4} f(x) = f(4)$.

$\Rightarrow f(x)$ is continuous at $x=4$.

<u>Case</u> 2: Continuity at any interior point $x = t$ in $[4, 7)$

Now,
$$\left.\begin{aligned} \lim_{x \to t^-} f(x) &= \lim_{x \to t^-} \sqrt{x-4} = \sqrt{t-4} \\ \lim_{x \to t^+} f(x) &= \lim_{x \to t^+} \sqrt{x-4} = \sqrt{t-4} \end{aligned}\right\} \lim_{x \to t} f(x) = \sqrt{t-4}$$

& $f(t) = \sqrt{t-4}$, so $f(x)$ is Cont. at any interior point of $[4, 7)$

Functions that are continuous everywhere:

① $f(x)$ = polynomial in x (say, x^5+7x+5)

② $f(x) = |x|$ ⤳ the absolute value function.

③ $f(x)$ = exponential functions $\left(e^x,\ e^{-(x+4)}\right)$

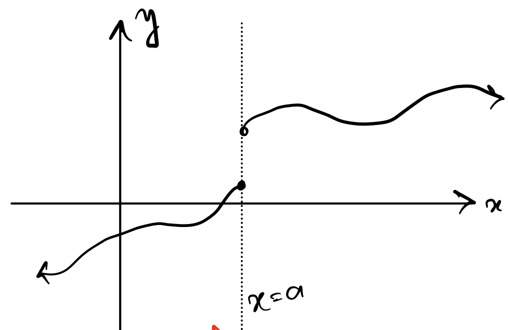
④ $f(x) = \sin x \,/\, \cos x$.

* $\tan x$, $\csc x$, $\sec x$, $\cot x$ are continuous on their respective domains.

$\log_a x$, $\ln x$, $\sqrt{x}$, $\sqrt[n]{x}$, rational functions are also continuous on their respective domains.

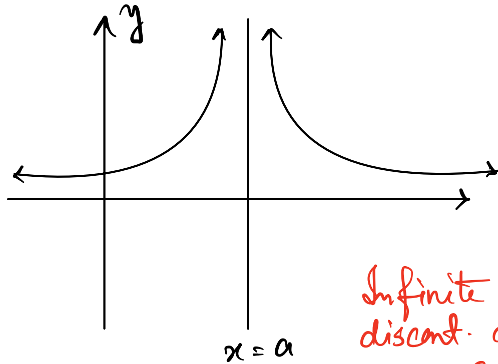
Discontinuities

Situation I. $\lim_{x\to a} f(x)$ DNE.



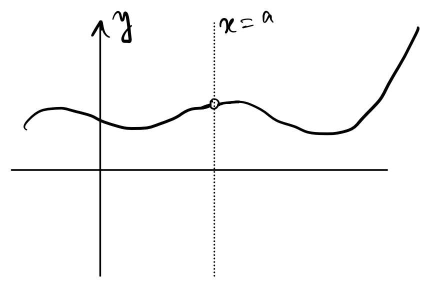
Jump Discont. at $x=a$

Situation 2. $\lim_{x \to a} f(x)$ exists, but not finite



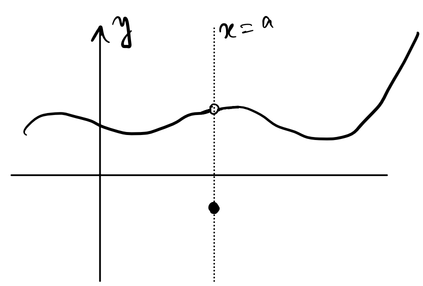
Infinite discont. at $x = a$

Situation 3. $\lim_{x \to a} f(x)$ exits, but $f(a)$ doesn't exist.



Removable discontinuity at $x = a$

Situation 4. Both $\lim_{x \to a} f(x)$ & $f(a)$ exists, but not equal.



Removable discontinuity at $x = a$

So there are three type of Discontinuity:

1. Infinite, when $\lim_{x \to a^{\pm}} f(x) = \pm\infty$
2. Removable, define $f(a)$ intelligently.
3. Jump, when $\lim_{x \to a^{-}} f(x) \neq \lim_{x \to a^{+}} f(x)$

Eg. $f(x) = \dfrac{x^2 - 2x}{x^2 - 4} = \dfrac{x(x-2)}{(x+2)(x-2)}$

Domain $= \{x \mid x \neq -2, 2\}$

At $x = -2$,

$\lim_{x \to -2} f(x) = \pm\infty \Rightarrow$ Infinite discont. at $x = -2$

At $x = 2$, $\lim_{x \to 2} f(x) = \frac{1}{2}$, but it is not defined, so $f(2)$ DNE.

$\Rightarrow$ Removable discont at $x = 2$

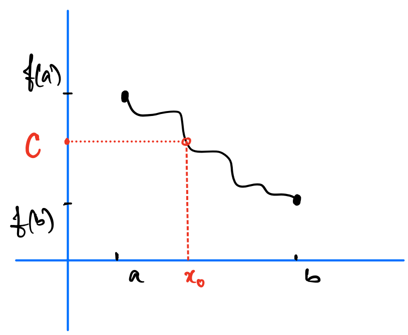
Note:- If $f(x)$ is cont at $x = a$, then the limits exists at $x = a$.

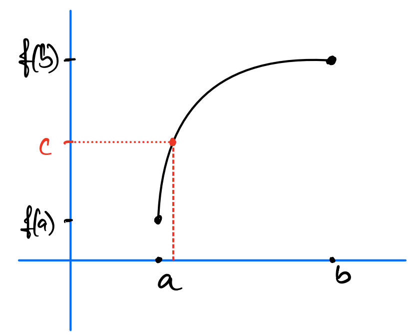
But if the limit exists at $x = a$, then it might or might not be continuous at $x = a$.

Intermediate Value Theorem (IVT):

If f is continuous on $[a,b]$ [ie, you can draw a continuous curve joining $(a, f(a))$ & $(b, f(b))$], then if c is any number between $f(a)$ & $f(b)$.

Then there is atleast one x_0 between a & b with $f(x_0)=c$.





Q. Is there any such value between 0 & $\pi/2$, for which $x=\cos x$ holds?

Sol^n: Let us consider $f(x)=\cos x - x$

Note $f(x)$ is Continuous on $[0, \frac{\pi}{2}]$.

Now, $f(\pi/2) = \cos(\pi/2) - \pi/2 = 0 - \pi/2 = -\pi/2$

& $f(0) = \cos(0) - 0 = 1 - 0 = 1$

Since, $-\pi/2 < 0 < 1 \Rightarrow f(\pi/2) < 0 < f(0)$

So by IVT, there exits $\theta \in (0, \pi/2)$ such that $f(\theta)=0$

ie, $\cos\theta - \theta = 0 \Rightarrow \cos\theta = \theta$.