Substitution :-

Recall: The chain rule gives:

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) \cdot g'(x).$$

Then antiderivative of $f'(g(x)) \cdot g'(x)$ is $f(g(x))$.

ie, $$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + \text{Constant.} \quad \text{(1)}$$

Let's take $\boxed{u = g(x)}$ $\quad (*)$

Then, $u' = g'(x)$, ie, $\frac{du}{dx} = g'(x)$

$$\Rightarrow \boxed{du = g'(x)\, dx} \quad (**)$$

Now, substituting $(*)$ & $(**)$ in (1) we get,

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C \quad \text{as}$$

$$\int f'(u)\, du = f(u) + C$$

Steps:
1. Choose proper substitution for u.
2. Convert the problem in terms of u.
3. Compute the integrals with respect to u.
4. Return back the substitutions.

Eg (1). Compute $\int (7x+9)^8\, dx$.

Let, $u = 7x+9$

$\therefore du = 7\, dx \Rightarrow dx = \frac{1}{7}\, du$

So, $\int u^8 \cdot \frac{1}{7} du = \frac{1}{7}\int u^8 du$

$$= \frac{1}{7} \cdot \frac{u^9}{9} + C$$

$$= \frac{u^9}{63} + C$$

$$= \frac{(7x+9)^9}{63} + C$$

Therefore, $\int (7x+9)^8 \, dx = \frac{(7x+9)^9}{63} + C$.

Eg (2). Compute $\int \frac{7}{7x+5} dx$

Let, $u = 7x+5 \Rightarrow du = 7\,dx$

$\Rightarrow dx = \frac{1}{7} du$

$$\int \frac{7}{7x+5} dx = \int \frac{7}{u} \cdot \frac{1}{7} du = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|7x+5| + C$$

Eg (3). Compute $\int \cos^2\theta \sin\theta \, d\theta$

$u = \cos\theta \Rightarrow du = -\sin\theta \, d\theta \Rightarrow -du = \sin\theta \, d\theta$.

$$\int \cos^2\theta \sin\theta \, d\theta = \int u^2 (-du)$$

$$= -\int u^2 du$$

$$= -\frac{u^3}{3} + C$$

$$= -\frac{\cos^3\theta}{3} + C$$

For function $y = f(x)$, the differential of y is $dy = f'(x)\,dx$

Substitution on Definite Integral:

Find $\int_0^1 \frac{3}{\sqrt{6x+5}}\,dx$

Method (1). First compute $\int \frac{3}{\sqrt{6x+5}}\,dx$ (without the constant)

Let $u = 6x+5 \Rightarrow du = 6\,dx \Rightarrow dx = \frac{1}{6}du$

Now, $\int \frac{3}{\sqrt{6x+5}}\,dx = \int \frac{3}{\sqrt{u}} \cdot \frac{1}{6}\,du = \int \frac{1}{2} u^{-1/2}\,du$

$$= \frac{1}{2} \frac{u^{1/2}}{1/2}$$

$$= u^{1/2}$$

$$= \sqrt{6x+5}$$

Then, $\int_0^1 \frac{3}{\sqrt{6x+5}}\,dx = \left[\sqrt{6x+5}\right]_{x=0}^{1}$

$$= \sqrt{6(1)+5} - \sqrt{6(0)+5}$$

$$= \sqrt{11} - \sqrt{5}$$

Method (2). Changing Integral limits.

Let $u = 6x+5 \Rightarrow du = 6\,dx \Rightarrow dx = \frac{1}{6}du$

&

x	0	1
u	$6(0)+5 = 5$	$6(1)+5 = 11$

Then, $\int_{x=0}^{1} \frac{3}{\sqrt{6x+5}}\,dx = \int_{u=5}^{11} \frac{3}{\sqrt{u}} \cdot \frac{1}{6}\,du = \frac{1}{2}\int_5^{11} u^{-1/2}\,du$

$$= \frac{1}{2} \cdot \frac{u^{1/2}}{1/2}\Big|_5^{11} = \sqrt{11} - \sqrt{5}$$